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Non-linear spin dynamics in ferromagnetic films and Schrödinger's equation in the vicinity of the zero-dispersion point

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Abstract. The non-linear spin dynamics in ferromagnets described by the non-linear Schrödinger equation with dispersion of the third order is investigated. The scenarios of evolution of various initial distributions of magnetization and the process of small-amplitude modulation of stable spatially non-localized waves are studied analytically and by computer simulation. The possibility of the formation of 'dark' and 'bright' quasi-soliton patterns and their stability are discussed.

1. Introduction

It is known that the basic peculiarities of the *linear* dynamics of spin excitations in magnets, namely the spectrum, amplitudes of oscillations, and conditions for excitations, are governed by the interactions that determine the ground state of the system. The long-range magnetic dipole–dipole interaction leads to only a minor perturbation in the ground state, but, nevertheless, its role in the spin dynamics is exceedingly great. It causes spatial dispersion of the long-wavelength range of the excitation spectrum and results in the peculiarities of the *non-linear* behaviour related to the shape of the *linear* dispersion law curve.

Analysis of the linear spin-wave spectrum of a normally magnetized isotropic ferromagnetic slab with free spins on the surface leads to a remarkable conclusion. There exists an interval of the wavenumbers for activation of the low-energy branch of the exchange-dipole spectrum wherein the second derivative of the dispersion law $\partial_k^2\omega(k)$ ($\omega(k)$ stands for the dispersion law; k is the wavenumber) is much less than the third derivative and may even go to zero (at the zero-dispersion point) [1]. The existence of such an interval is not caused by the hybridization of the adjacent branches of the spin-wave spectrum, but is determined completely by the competition of dispersions of two types—an exchange one and a magnetostatic one. The region of zero dispersion may occur not only in the case of exchange-dipole waves, but also for other types of magnetic excitation (e.g. surface magnetostatic waves (see [2])). Note that the first observations of 'dark' solitons of magnetostatic surface waves in thin magnetic films were presented in [3]. The treatment of the weakly non-linear dynamics in terms of the traditional model—using the non-linear Schrödinger equation (NSE)—fails, because the role of the highest derivatives with respect to the spatial coordinate (in particular that of the third order) now becomes important in the evolution equation for the envelope of spin waves. At

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the present time are a number of papers available that are devoted to the analysis of the NSE with high-order dispersion (the generalized NSE). The possible types of such equations in non-linear optics and their integrability have been discussed in [4, 5]. A detailed analysis of optical soliton and quasi-soliton transport, and the equilibrium conditions with respect to the Cherenkov radiation, has been carried out in [5]. Against the background of the generalized NSE, the process of automodulational amplification of the noise of spin oscillations in the travelling magnetostatic wave was studied in [6].

The present paper is devoted to the analysis of the features of the non-linear spin-wave processes in ferromagnetic films in the vicinity of the zero-dispersion point. In section 2 the basic equations for the weakly non-linear dynamics of exchange-dipole waves and the conservation laws related to them are considered. The low-amplitude long-wavelength modulations of spatially non-localized non-linear waves are discussed in section 3, where the 'dark solitons' are predicted also. In section 4 the peculiarities of the evolution of the spatially localized initial distributions of the magnetization are analysed. The long-living soliton-like patterns, travelling with nearly constant velocity, may arise from these distributions after the 'splitting off' of the non-linear wave sequence has occurred. Such 'bright' quasi-solitons leave behind a static low-amplitude tail when travelling and radiate forward the low-amplitude waves. The shape of the 'bright' soliton and the long-wavelength modulations of its tail are described analytically. In section 5 the variation of the structure of the bright soliton in the process of motion is analysed. 'Cherenkov-type' radiation is considered as a mechanism of change of the soliton structure. The non-linear wave sequence 'split off' after quasi-soliton formation is discussed in section 6.

2. The equation for the weakly non-linear dynamics of exchange-dipole waves in the zero-dispersion range; conservation laws

Consider an isotropic ferromagnetic slab magnetized along its normal direction (z-axis) with free spins on its surfaces. We shall use the version of the reductive perturbation theory proposed in [1] to deduce effective equations for the weakly non-linear dynamics of the activation exchange-dipole low-energy branch of the wave spectrum. Unlike in [1], we are interested in the zero-dispersion range, where $\partial_k^2\omega(k) \approx 0$. We have found that the dispersion effects and the non-linear effects reach a balance for the following choice of slow variables: $X = \varepsilon^2(x + c_g t)$, $\tau = \varepsilon^6 t$, where t and x are time and space coordinates, respectively, ε is a small parameter specifying the deviation from equilibrium, and c_g is the group velocity of the spin waves. The following evolution equation is derived according to this choice of slow variables after calculations similar to those carried out in [1]:

$$\mathrm{i}\,\partial_\tau\varphi + \frac{\mathrm{i}}{6}\,\partial_k^3\omega(k)\,\partial_X^3\varphi + g|\varphi|^2\varphi = 0. \tag{1}$$

Note that this choice of the variables leads to $|\varphi| \approx \mathrm{O}(\varepsilon^3)$. All of the quantities in (1) are dimensionless:

$$\tau \to \omega_H\tau \qquad X \to X/d$$
$$k \to kd \qquad \omega(k) \to \omega(k)/\omega_H$$
$$g \to g/\omega_H.$$

Here d is the thickness of a slab (film), $\omega_H = \gamma H_0$, γ is the gyromagnetic ratio, and H_0 is the external field. The parameter g, characterizing the interaction of the waves, in the general case depends on the wavenumber and the slab parameters, and coincides with that mentioned in [1]. The relation between the magnetization in the slab plane and the field φ is [1]

$$M_0^{-1}(m_x + \mathrm{i}m_y) = (1 + \omega(k)/\omega_0(k))\varphi(X,\tau) \tag{2}$$

where

$$\omega^2(k) = \omega_0(k)[\omega_0(k) + \omega_M \sigma(k)]$$
$$\sigma(k) = 1 - [1 - \exp(-|kd|)]/|kd|$$
$$\omega_0(k) = \omega_H + \gamma M_0 \alpha k^2$$
$$\omega_M = 4\pi\gamma M_0.$$

M_0 is the saturation magnetization and α is the exchange constant. Note that in our case the coefficient $1 + \omega/\omega_0$ changes negligibly and is close to 2 over a rather wide range of k-values (see [1]).

We present now conservation laws that are connected with equation (1). It can easily be verified that the evolution of any spatially localized distribution of the field φ occurs in such a way that

$$\int_{-\infty}^{\infty} |\varphi|^2 \, \mathrm{d}X = N = \text{constant}. \tag{3}$$

Equation (1) can be deduced with the help of the Lagrange-function density

$$L = \frac{2}{\gamma M_0}\left[\mathrm{i}(\varphi\,\partial_\tau\varphi^* - \varphi^*\,\partial_\tau\varphi) + \frac{\mathrm{i}}{6}\,\partial_k^3\omega\,(\varphi\,\partial_X^3\varphi^* - \varphi^*\,\partial_X^3\varphi) - g|\varphi|^4\right].$$

The expression for the energy of the system E (see [1]) in terms of the field φ is as follows:

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \mathrm{d}X \left[\frac{\partial L}{\partial(\partial_\tau\varphi)}(\partial_\tau\varphi) + \frac{\partial L}{\partial(\partial_\tau\varphi^*)}(\partial_\tau\varphi^*) - L\right] \\ &= \frac{2}{\gamma M_0}\int_{-\infty}^{\infty} \mathrm{d}X \left[\frac{\mathrm{i}}{6}\,\partial_k^3\omega\,(\varphi^*\,\partial_X^3\varphi - \varphi\,\partial_X^3\varphi^*) + g|\varphi|^4\right] = \text{constant}. \end{aligned} \tag{4}$$

The field momentum of the system is defined by the expression

$$\begin{aligned} P &= \int_{-\infty}^{\infty} \mathrm{d}X \left[\frac{\partial L}{\partial(\partial_\tau\varphi)}(\partial_X\varphi) + \frac{\partial L}{\partial(\partial_\tau\varphi^*)}(\partial_X\varphi^*)\right] \\ &= \mathrm{i}\frac{2}{\gamma M_0}\int_{-\infty}^{\infty} \mathrm{d}X\,(\varphi\,\partial_X\varphi^* - \varphi^*\,\partial_X\varphi) = \text{constant}. \end{aligned} \tag{5}$$

Conservation laws (3)–(5) are very useful in the analysis of the evolution of initial-field (φ-) distributions (see sections 4 and 5).

3. Long-wavelength modulations of monochromatic non-linear waves

In order to specify the role of the third-order dispersion terms in equation (1), let us consider the stability conditions for the plane wave which is an exact solution of (1):

$$\varphi_0 = (w_0)^{1/2}\exp[\mathrm{i}(\Omega_0\tau + k_0 X)]. \tag{6}$$

Here, w_0 is the amplitude, and k_0 is the wavenumber corresponding to the deviation of the wave vector from the value at the point at which the second derivative of the spin-wave dispersion law goes to zero. The dispersion law for the wave (6) has the form

$$\Omega_0 = \frac{1}{6}\,\partial_k^3\omega\,k_0^3 + g w_0 \tag{7}$$

where the value of $\partial_k^3\omega$ is calculated at a point where $\partial_k^2\omega(k) = 0$. Let us present the solution of equation (1) in a form that allows us to establish the stability conditions: $\varphi = \varphi_0(1 + \tilde{\varphi})$, where $|\tilde{\varphi}| \ll 1$ and

$$\tilde{\varphi} = \tilde{\varphi}_1 \exp[\mathrm{i}(pX + \lambda\tau)] + \tilde{\varphi}_2 \exp[-\mathrm{i}(pX + \lambda^*\tau)].$$

By means of linear analysis of the stability, we reach the conclusion that the plane wave (6) is unstable relative to long-wavelength perturbations in view of the inequality $gk_0\,\partial_k^3\omega < 0$. In particular, for $g, \partial_k^3\omega > 0$ (for the sake of definiteness, these conditions are assumed to be realized in the rest of the calculations), waves with $k_0 < 0$ are modulationally unstable. If in this case $\Omega_0 > 0$, that is $gw_0 > \frac{1}{6}\,\partial_k^3\omega\,k_0^3$, then modulationally unstable waves propagate to the right.

Note that the static wave with $\Omega_0 = 0$, $k_0 = -(6gw_0/\partial_k^3\omega)^{1/3} < 0$, belongs to the domain $k_0 < 0$. From the viewpoint of linear analysis, it is unstable (the increment of the amplitude is proportional to $|p|\surd\{|gk_0\,\partial_k^3\omega|w_0\}$). However, computer simulation leads to an interesting conclusion. The low-amplitude static wave which has been forming over a long period of time at the rear of the 'bright quasi-soliton' (see below) undergoes only weak modulations over a long distance. Apparently, the 'modulated structure' (6) with $\Omega_0 = 0$ is realized in the above-mentioned spatial interval, since its energy per period (4) is less than that for the $\varphi = 0$ state, and therefore non-linear interactions may stabilize the static wave under the given definite conditions.

At the same time, it follows from the analysis of the linearized problem that, against the background of stable (for $k_0 > 0$) waves, propagating to the left, Goldstone modes may exist, with dispersion laws $\lambda_i(p)$, $i = 1, 2$:

$$\lambda_i(p) = \left[\frac{1}{2}\,\partial_k^3\omega\,k_0^2 + \varepsilon_i m\right] p + \left[\frac{1}{6}\,\partial_k^3\omega + \varepsilon_i n\right] p^3 \tag{8}$$

where $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, $m^2 = gk_0w_0\,\partial_k^3\omega$, and $n = (\partial_k^3\omega\,k_0)^2/8m$. In deriving (8), we have assumed that $(k_0d)(pd)^2 \ll 4(gw_0/\partial_k^3\omega)$.

We shall describe now the long-wavelength low-amplitude modulations of the wave (6). The corresponding solutions of equation (1) are presented in the form

$$\varphi = \sqrt{w}\,\exp[\mathrm{i}(\Omega_0\tau + k_0X + \chi)] \tag{9}$$

where $w = w_0(1 + \nu(X, \tau))$, $|\nu| \ll 1$, $k_0 \gg \partial_X\chi$, and $|\Omega_0| \gg \partial_\tau\chi$. We shall write down the set of equations for the fields ν and χ to within the squared terms in ν and χ. The derivatives with respect to spatial coordinates of order higher than the fourth may be neglected in the long-wavelength limit in terms linear in ν and χ in these equations. Only low-order derivatives will be retained in the non-linear terms. As a result we obtain the following set of equations:

$$\begin{aligned} &\mathbf{L}\nu - \partial_k^3\omega\,k_0\,\partial_X^2\chi - \frac{1}{2}\,\partial_k^3\omega\,\partial_X[(\partial_X\chi)^2 + 2k_0\nu\,\partial_X\chi] = 0 \\ &w_0\mathbf{M}\nu + \mathbf{L}\chi - \frac{1}{2}\,\partial_k^3\omega\,k_0(\partial_X\chi)^2 = 0 \end{aligned} \tag{10}$$

where

$$\begin{aligned} \mathbf{L} &= \mathbf{L}_0 + \frac{1}{6}\,\partial_k^3\omega\,\partial_X^3 \\ \mathbf{L}_0 &= \partial_\tau - \frac{1}{2}\,\partial_k^3\omega\,k_0^2\,\partial_X \\ \mathbf{M} &= -g + \frac{1}{4}w_0^{-1}\,\partial_k^3\omega\,k_0\,\partial_X^2. \end{aligned}$$

The field ν may be easily eliminated from (10). The long-wavelength-limit relation between ν and χ in the principal approximation is used to transform the non-linear terms:

$$w_0g\nu = \mathbf{L}_0\chi. \tag{11}$$

After simple calculations, closed equations for $\chi(X, \tau)$ were obtained:

$$\left\{-\mathbf{L}_0^2 - \frac{1}{3}\,\partial_k^3\omega\,\mathbf{L}_0\,\partial_X^3 + \partial_k^3\omega\,k_0 g w_0\,\partial_X^2 - \left(\frac{\partial_k^3\omega}{2}k_0\right)^2\partial_X^4\right\}\chi$$
$$+\frac{\partial_k^3\omega}{2}\,\partial_X\left[g w_0(\partial_X\chi)^2 + 2k_0\,\partial_X\chi\,\mathbf{L}_0\chi\right] + \frac{\partial_k^3\omega}{2}k_0\mathbf{L}_0(\partial_X\chi)^2 = 0. \tag{12}$$

In deducing (12), we neglect terms that are cubic in the fields ν and χ in the dynamic equations. Analysis shows that this is appropriate for waves with stationary profiles, travelling with the velocity u, if

$$|g w_0 + 3k_0 l| \gg |lA|. \tag{13}$$

Here $l = u - \frac{1}{2}\,\partial_k^3\omega\,k_0^2$, and A is a typical amplitude of the field $\partial_X\chi$. If the soliton exists, $A \sim |\gamma_0/\delta|$ (see below).

Let us consider solutions of equation (12) of the type representing waves with stationary profiles:

$$\chi = \chi(\xi) \qquad \partial_\xi\chi = f(\xi) \qquad \xi = X + u\tau. \tag{14}$$

Then $f(\xi)$ is defined by the equation

$$(\partial_\xi f)^2 = \delta f^3 + \gamma_0 f^2 + \beta f + \alpha \tag{15}$$

where $\delta = \frac{1}{3}\,\partial_k^3\omega\, r(g w_0 + 3k_0 l)$, $\gamma_0 = r(m^2 - l^2)$, $r^{-1} = \frac{1}{3}\,\partial_k^3\omega\, l + 2mn$, and α and β are arbitrary constants. Let us analyse possible solutions of equation (15).

For $\alpha = \beta = 0$ and with the inequality

$$\gamma_0 > 0 \tag{16}$$

fulfilled, equation (15) allows the formation of solitary waves. The corresponding solution has the form

$$f = -\frac{\gamma_0}{\delta}\operatorname{sech}^2\left(\frac{1}{2}\gamma_0^{1/2}\xi\right). \tag{17}$$

The results of computer simulations of the solution of equation (1) with impulse excitation of the inhomogeneous state against the background of harmonic wave (6) are presented in figure 1. The long-living localized structure arises in the process of evolution, and its profile is approximated rather well by the solution (17) (see the dotted curve in figure 1).

In the case under consideration, inequality (13) takes the form

$$|\partial_k^3\omega\,(g w_0 + 3k_0 l)^2| \gg 3|l(m^2 - l^2)|. \tag{18}$$

In addition to (18), the condition for the long-wavelength approximation must be fulfilled (the typical size of the soliton $\kappa^{-1} \sim 1/\sqrt{\gamma_0}$ must be greater than the wavelength k_0^{-1}):

$$\left|\frac{\kappa}{k_0}\right| \sim \left|\frac{\sqrt{\gamma_0}}{k_0}\right| \ll 1. \tag{19}$$

When the wave (6) is stable relative to linear perturbations, $w_0\,\partial_k^3\omega\,k_0 g \equiv m^2 > 0$ ($\partial_k^3\omega > 0$, $k_0 > 0$, $g > 0$ for definiteness), inequalities (18), (19) will be necessarily fulfilled for the localized perturbations, which spread to the left against the background of the wave (6), if

$$0 < m - l \ll \min\left\{\frac{\partial_k^3\omega}{6m^2}(g w_0 + 3k_0 m)^2, k_0^2\left(n + \frac{1}{6}\,\partial_k^3\omega\right)\right\}. \tag{20}$$

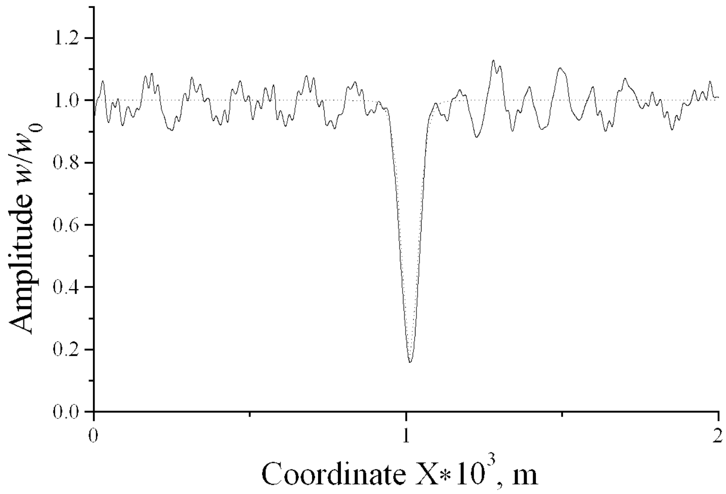

Figure 1. A 'dark quasi-soliton' against the background of a non-linear harmonic wave. The dotted curve corresponds to the exact solution of (17) for $(k_0 d) \approx 10^{-1}$, $\partial_k^3 \approx 4 \times 10^{-2}$, $g \approx 0.9$, and $d \approx 1\ \mu$m; the length of the plate in the direction of wave propagation is $\sim$0.5 cm.

In this case the inverse width of the soliton, κ, and its amplitude, A, are defined by the expressions

$$\kappa^2 = \gamma_0 \approx \left(m + \frac{1}{2}\,\partial_k^3\omega\, k_0^2 - u\right)\left(n + \frac{1}{6}\,\partial_k^3\omega\right) > 0$$

$$A = \frac{\gamma_0}{\delta} \approx \kappa^2 m(\partial_k^3\omega + 6n)\left[\partial_k^3\omega\,(g w_0 + 3k_0 m)\right]^{-1}.$$

Using material parameters presented in [1], namely, at the point of zero dispersion, $\partial_k^3\omega \approx 4 \times 10^{-2}$, and choosing $k_0 d \approx 10^{-2}$, $\omega_H \approx 10^{10}\ \mathrm{s}^{-1}$, and $d \approx 1\ \mu$m, we can show that the above-mentioned inequalities are fulfilled for these values. Besides this, the conditions of φ being small and $k_0 \gg \partial_X \chi$ are also fulfilled.

Returning to the Goldstone modes (8), note that in this case, against the background of the wave (6), the modes with $\varepsilon_1 = 1$ interact most intensively, as can easily be shown. In the case where the modulation of the wave (6) is due only to the excitation and interaction of these modes, it is convenient to pass over to the frame of reference moving with phase velocity $v_{01} = \frac{1}{2}\,\partial_k^3\omega\, k_0^2 + m$ for these modes, and to use the new variables $\xi = X + v_{01}\tau$ and $T \equiv \tau$ instead of X and τ. As long as the dependence of the field $\chi(\xi, T)$ on the variable T is weak in the case under consideration, we may neglect terms of the order of $\partial_T^2\chi$ and $\partial_T\,\partial_\xi^3\chi$ in equations (12), and also the non-linear terms containing the derivatives $\partial_T\chi$. As a result, equation (12) is reduced to the Korteweg–de Vries (KdV) equation

$$\partial_T f + \left(\frac{1}{6}\,\partial_k^3\omega + n\right)\partial_\xi^3 f - \frac{\partial_k^3\omega}{4}\left(3k_0 + \frac{g w_0}{m}\right)\partial_\xi f^2 = 0 \tag{21}$$

where $f = \partial_\xi\chi$. It can easily be verified that solution (19) generalizes the one-soliton solution of the model (21). The speed of the soliton in the model (21) is lower than the speed of Goldstone's waves belonging to branch (8) with $\varepsilon_1 = 1$. The closer the soliton speed to the phase velocity of the Goldstone mode, the better the likelihood of inequality (20) being fulfilled, and the greater the soliton amplitude and the lower its velocity.

Note that according to (11) the modulation of the wave (6) is synchronized with the modulation of the wave vector.

4. Peculiarities of the evolution of the spatially localized initial distributions of the magnetization field

As computer simulations show, the transformation of the spatially localized initial perturbation of the field $\psi = a\varphi(bX)$, with $a = g^{1/2}$ and $b = (\frac{1}{6}\,\partial_k^3\omega)^{1/3}$, which can be chosen, for instance, in the form‡ $\psi(X, \tau = 0) = \sqrt{2}\,\mathrm{sech}(X - X_0)$, occurs through the following scenario. At first, the $|\psi|^2$-distribution is strongly deformed and radiates a non-linear sequence of waves to the left (see figure 2). After part of the energy has been ‘thrown off’, there remains a long-living localized pattern, which moves with nearly constant velocity from the right to the left. The change in shape of such ‘quasi-soliton’ happens in two ways: firstly, at the expense of low-amplitude waves, by forward radiation (to the left); and secondly, by leaving behind the low-amplitude tail, which presents a nearly static harmonic wave with $|\psi|^2 \approx$ constant at a sufficiently long distance to the rear of the ‘quasi-soliton’ (see figure 3). It must be emphasized that the width of the wave packet of the initial distribution of the field ψ should be taken in such a way that the wavenumbers of the wave-packet harmonics do not violate the inequality $\partial_k^2\omega \ll \partial_k^3\omega$.

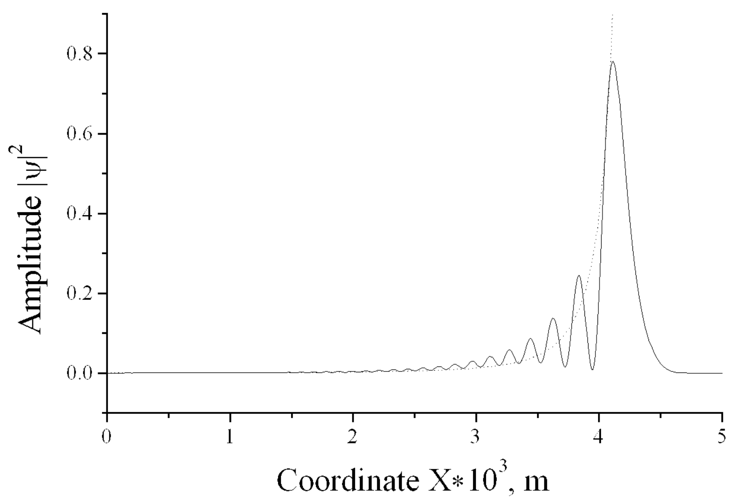

Figure 2. ‘Splitting off’ the non-linear wave sequence from the initial distribution of the magnetization.

The processes mentioned above can be understood qualitatively on the basis of the conservation laws (3)–(5). For further analysis, it is convenient to write down the field φ in the form

$$\varphi = R \exp \mathrm{i}\theta. \tag{22}$$

Since, according to (3),

$$\int_{-\infty}^{\infty} R^2\,\mathrm{d}X = N = \text{constant}$$

one can introduce the notion of the centre of gravity for the distribution $|\varphi|^2 = R^2$:

$$\langle X\rangle = \frac{1}{N}\int_{-\infty}^{\infty} XR^2\,\mathrm{d}X. \tag{23}$$

‡ Equation (1) in terms of ψ has the form $\mathrm{i}\psi_\tau + \mathrm{i}\psi_{XXX} + |\psi|^2\psi = 0$.

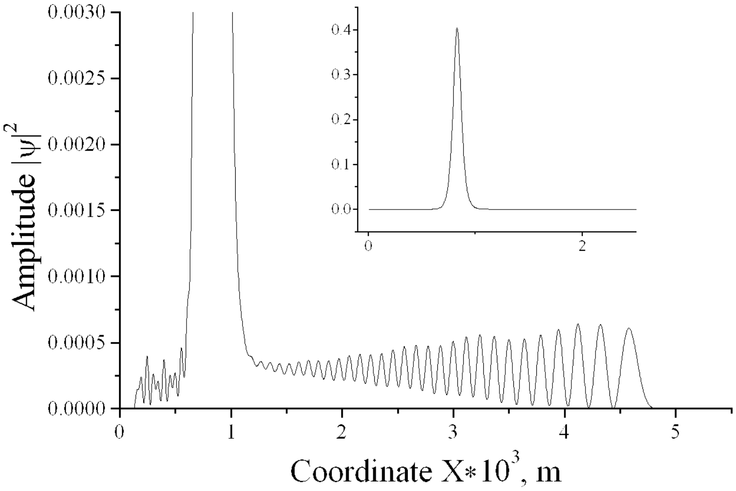

Figure 3. The 'bright quasi-soliton' moving to the left (in the inset). The quasi-static 'tail' (left) and small-amplitude forward radiation (right) are observed at the foot of a peak after zooming in.

Using (1), it can easily be shown that the speed of the centre of gravity has the form [7]

$$v = \frac{\mathrm{d}}{\mathrm{d}\tau}\langle X\rangle = -\frac{\partial_k^3\omega}{N}\int_{-\infty}^{\infty}\mathrm{d}X\,[(\partial_X R)^2 + (\partial_X^2\theta)^2 R^2]. \tag{24}$$

The direction of motion depends on the sign of $\partial_k^3\omega$ (in our case the sign is positive, so it moves to the left). The speed of the centre of gravity is determined by the gradients of the phase and the amplitude of the field φ, as follows from (24). The conservation law for the field momentum (5) also gives useful information.

Let the initial distribution have no phase gradient, and let the φ-function be, for instance, real valued; then $P = 0$. After separation of the wave sequence from this distribution, the function R^2 represents a two-peak curve along the x-axis. From the condition of field momentum conservation $P = 0$ (see (5)), it follows that 'mean' wavenumber corresponding to the wave sequence has to differ in sign from the 'mean' wavenumber corresponding to the soliton. Since only waves travelling to the left with positive wavenumbers are modulationally stable, the 'mean' wavenumber localized on the 'soliton' must be negative. It should be noted that in this case the 'soliton' and small-amplitude radiation move in one direction and this does not contradict the momentum conservation law.

As was pointed out earlier, the energy estimate allows one to explain qualitatively the tendency towards formation of a static quasi-monochromatic tail of the 'soliton'. Such a tail corresponds to the following solution of equation (1):

$$\varphi = \sqrt{w_0}\exp\left[-\mathrm{i}X(6gw_0/\partial_k^3\omega)^{1/3} + \mathrm{i}\alpha\right] \tag{25}$$

where w_0 and α are real constants. The long-wavelength static modulations of the spatial structure (25) are described by equation (15) with $u = \Omega_0 = 0$, $k_0 = -(6gw_0/\partial_k^3\omega)^{1/3} < 0$. In this case, condition (16) fails, and, hence, there are no solitary waves against the background of the non-uniform state (25). However, for $\alpha > 0$ (see (15)), small-amplitude modulations of the wave (25) may exist, if the following inequality is satisfied:

$$D = \gamma^2 - 4\beta\delta > 0. \tag{26}$$

Here $\delta = -\frac{20}{3}k_0$ and $\gamma = -5k_0^2$. To make the corresponding solution of equation (15) continuous under change of the parameter β, we should take $\delta\beta < 0$. Under this condition, we find

$$f = a - (a-c)\mathrm{dn}^2(\eta, k) \qquad \eta = \frac{1}{2}X\sqrt{(a-c)\delta} \qquad k = \sqrt{\left|\frac{c}{c-a}\right|} \tag{27}$$

where $a = (\sqrt{D} - \gamma)/\delta > 0$ and $c = (-\gamma - \sqrt{D})/\delta < 0$. The cnoidal wave (27) describes the long-wavelength modulations of the tail of the 'bright soliton'. The amplitude of the modulation rises for movement far from the soliton. It is possible that this situation can be described, if slow dependence of the spatial coordinate of the parameters of the cnoidal wave is assumed, and if the Whithams [8] method of averaging is used.

It follows from the results of the computer simulation that in the region of 'soliton' localization the wave vector k_0 ($k_0 < 0$) is nearly constant, and also the size of the 'soliton' Δ is greater than the wavelength of the modulation of its profile ($\Delta > |k_0|^{-1}$). This information can be used in the construction of the approximate solution of equation (1), which will describe the internal structure of the 'quasi-soliton'. The following procedure for the construction of the asymptotic expansion in powers of the parameter $|k_0|^{-1}\Delta^{-1} \ll 1$ corresponds to the version of the non-linear perturbation theory:

$$\varphi = R(\zeta)\exp\left\{\mathrm{i}\left[k_0 X + \frac{\partial_k^3\omega}{6}k_0^3\tau + \sum_{n=1}^{\infty}\varepsilon^n\Omega^{(n)}\tau + \chi(\zeta)\right]\right\} \tag{28}$$

where ε is a small parameter, characterizing slow evolution of the wave shape, and

$$\begin{aligned}
\zeta &= \varepsilon(X + \nu\tau)\\
\nu &= \nu^{(0)} + \sum_{n=1}^{\infty}\varepsilon^n\nu^{(n)}\\
R(\zeta) &= \sum_{n=1}^{\infty}\varepsilon^n R^{(n)}(\zeta)\\
\chi &= \sum_{n=1}^{\infty}\varepsilon^n\chi^{(n)}(\zeta).
\end{aligned}$$

The equations arising in every order of ε are consequently solved. Here the functions $\chi^{(i)}(\xi)$ are expressed in terms of $R^{(i)}(\xi)$, and a closed equation for $R^{(1)}(\xi)$ is obtained:

$$\Omega^{(2)}R^{(1)} + \frac{1}{2}\,\partial_k^3\omega\,k_0\,\partial_\zeta^3 R^{(1)} - g(R^{(1)})^3 = 0. \tag{29}$$

Among its bounded solutions, both localized and periodic non-linear waves exist. The integration constants, and parameters $\Omega^{(n)}$ and $\nu^{(n)}$, for both types of solution of the equation for $R^{(1)}$, may be chosen in such a way that no secular terms, caused by the phase shift of the wave and by its translation, proportional to $R^{(1)}$ or $\partial_\zeta R^{(1)}$ will appear. For the self-localized wave, after simple calculations we found

$$\begin{aligned}
&\nu^{(0)} = \frac{1}{2}\,\partial_k^3\omega\,k_0^2 && \nu^{(2)} = -\frac{1}{6}\,\partial_k^3\omega\,\kappa^2\\
&\Omega^{(2)} = \frac{1}{2}|\partial_k^3\omega\,k_0|\kappa^2 && R^{(2)} = \nu^{(1)} = \Omega^{(1)} = \Omega^{(3)} = \Omega^{(4)} = 0\\
&R^{(1)}(\zeta) = \kappa\sqrt{\left|\frac{\partial_k^3\omega\,k_0}{g}\right|}\,\mathrm{sech}(\kappa\zeta) && \chi^{(1)}(\zeta) = -\frac{\kappa}{2k_0}\tanh(\kappa\zeta)\\
&\frac{\Omega^{(2)}}{g} > 0 && g\,\partial_k^3\omega\,k_0 < 0.
\end{aligned} \tag{30}$$

Here κ is the parameter determining the size of the quasi-soliton ($\Delta \sim \kappa^{-1}$, $\kappa \ll |k_0|$).

The tendency towards formation of a soliton of the type (30) is explained by the fact that for $k_0 > 0$ we move away from the point of zero dispersion, where $\partial_k^2\omega = 0$, into a region where the ordinary NSE with quadratic dispersion allows the existence of exponential 'bright' solitons. That is why the result obtained, equations (30), is close to that presented in [9], where the term with anomalous dispersion was treated as a small perturbation to the ordinary NSE with quadratic dispersion. In the papers [9–11], it was established that the theoretical description of a small-amplitude radiation from a newly formed 'soliton' of the type (30) requires another version of the perturbation theory, similar to the VKB method.

5. Small-amplitude radiation from 'bright solitons'

Small-amplitude radiation from a 'bright quasi-soliton' (30) cannot be described by an expansion of the type given by (28) in powers of the parameter $\varepsilon = \mathrm{O}|\kappa/k_0|$, because it corresponds to terms with exponentially small amplitudes, that are non-analytic in ε. According to (28) and (30), the shape of the 'bright soliton' in the main approximation is

$$\varphi_{\mathrm{sol}} \approx \varphi_{\max}\,\mathrm{sech}(\kappa\zeta)\exp\mathrm{i}\theta \qquad \theta = k_0 X + \frac{1}{6}\,\partial_k^3\omega\,k_0^2\tau + \Omega^{(2)}\tau$$
$$\zeta = X + \nu\tau \qquad \nu = \frac{1}{2}\,\partial_k^3\omega\left(k_0^2 - \frac{1}{3}\kappa^2\right) \qquad \varphi_{\max} = \kappa\sqrt{\left|\frac{\partial_k^3\omega\,k_0}{g}\right|}. \tag{31}$$

Such a soliton cannot be stationary, because it interacts resonantly with the radiation wave, whose velocity coincides with the soliton velocity ν. Let us represent the radiation field in the form

$$\varphi_{\mathrm{rad}} \approx \varphi_{\max} f(\zeta,\tau)\exp\mathrm{i}\theta. \tag{32}$$

At large distances from the soliton ($\zeta \to -\infty$), the function $f(\zeta,\tau)$ satisfies the linear equation

$$\mathrm{i}\,\partial_\tau f + \mathrm{i}\nu^{(2)}\,\partial_\zeta f - \Omega^{(2)} f + \frac{\mathrm{i}}{6}\,\partial_k^3\omega\,\partial_\zeta^3 f - \frac{1}{2}\,\partial_k^3\omega\,k_0\,\partial_\zeta^2 f = 0. \tag{33}$$

The quantities $\Omega^{(2)}$ and $\nu^{(2)}$ are defined by (30). It can easily be shown by solving this equation that the following linear wave is coupled resonantly to the soliton (31):

$$f(\zeta,\tau) = C\exp(\mathrm{i}r_0\zeta)H(-\nu_g\zeta)H(|\nu_g\tau| - |\zeta|) \tag{34}$$

where $r_0 = -3k_0 > 0$ is the wave vector corresponding to the condition for the 'Cherenkov' resonance ($\Omega(r_0) = 0$; see below). The Heaviside step functions $H(z) = \frac{1}{2}(\mathrm{sgn}\,z + 1)$ in (34) determine the direction of propagation and the position of the radiation front, depending on the group velocity ν_g of the wave (32) in the frame of reference related to the soliton (31):

$$\nu_g = \left.\frac{\partial\Omega}{\partial r}\right|_{r=r_0} = \frac{\partial_k^3\omega}{6}(\kappa^2 + r_0^2) \qquad \Omega(r) = \frac{1}{6}\,\partial_k^3\omega\,(r - r_0)(r^2 + \kappa^2). \tag{35}$$

Here $\Omega(r)$ is the dispersion law for the linear waves (33) in the frame of reference related to the soliton. The non-analytic dependence of the amplitude of radiation C on the parameter ε in the main approximation in ε may be found by a method similar to the VKB one [9, 11]:

$$C = \frac{\Gamma}{\varepsilon}\exp[-\pi/2\varepsilon] \qquad \varepsilon = |\kappa/r_0| \ll 1 \tag{36}$$

where Γ is a complex parameter. The quantities k_0 and κ, characterizing the shape and functional dependence of the 'quasi-soliton' (31), gain a 'slow' dependence on time due to the radiation (34). We shall determine this dependence using conservation laws (3) and (5):

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = 0. \tag{37}$$

We get the following equations from (37):

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\kappa\left|\frac{\partial_k^3\omega\, k_0}{g}\right|\right\} \approx -\frac{1}{2}\left|\frac{\partial_k^3\omega\, k_0}{g}\right| |v_g||C_1|^2 \\
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\kappa\left|\frac{\partial_k^3\omega\, k_0}{g}\right| k_0\right\} \approx -\frac{1}{2}\left|\frac{\partial_k^3\omega\, k_0}{g}\right| |v_g||C_1|^2(r_0+k_0) \\
&C_1 = \Gamma r_0 \exp\left(-\frac{\pi r_0}{2|\kappa|}\right)
\end{aligned} \tag{38}$$

which coincide in form with ones presented in [12]. However, in [12] it was considered a problem that $\partial_k^2\omega \gg \partial_k^3\omega$, and hence, in the main approximation, the parameters v_g and r_0 are fixed. In the case under investigation, we are interested in the zero-dispersion region ($\partial_k^2\omega \approx 0$); therefore the parameters v_g and r_0 are governed by the dynamics of the problem (they are expressed in terms of the wave vector k_0, which changes in the process of radiation).

It can easily be shown that the equation set (38) has the motion integral

$$\kappa|k_0|^{4/3} = \sigma = \text{constant}. \tag{39}$$

Using (39) we get the temporal evolution for k_0 in the region where

$$3\varepsilon = |\kappa/k_0| = \sigma|k_0|^{-7/3} \ll 1$$

from the first equation of the set (38). As in [12], it is convenient to introduce the variable y:

$$y = \frac{3\pi}{\sigma}|k_0|^{7/3}$$

and use it instead of k_0. For the region under investigation, $y \gg 1$. Let $y(\tau = 0) = y_0$. It is obvious that $y(\tau) > y_0 \gg 1$ and $y > \frac{23}{7}\ln y_0$. By integrating the first of the equations of the set (38) by parts, we can express y in terms of τ with logarithmic accuracy:

$$\begin{aligned}
&y = y_0\left\{\mu(\tau) + \mathrm{O}(y_0^{-1}\ln\mu(\tau))\right\} \\
&\mu(\tau) = 1 + y_0^{-1}\ln\left[1 + \frac{\tau}{\tau_{\mathrm{ch}}}\right].
\end{aligned} \tag{40}$$

Here, a characteristic time is introduced:

$$\tau_{\mathrm{ch}} = \frac{4\pi}{63}\left(\frac{3\pi}{\sigma}\right)^{9/7} y_0^{-23/7}\frac{1}{\left|\Gamma^2\,\partial_k^3\omega\right|}\exp y_0.$$

Using (40), we find that, as a result of radiation, the soliton parameters change over time according to the relations

$$\begin{aligned}
&k_0 = k_0(\tau=0)[\mu(\tau)]^{3/7} \qquad && \kappa^{-1} = \kappa^{-1}(\tau=0)[\mu(\tau)]^{4/7} \\
&v = v(\tau=0)[\mu(\tau)]^{6/7} \qquad && |\varphi|_{\max} = |\varphi_{\max}(\tau=0)|[\mu(\tau)]^{-5/14}.
\end{aligned} \tag{41}$$

Expressions (41) show that over large time intervals $\tau \gg \tau_{\mathrm{ch}}$, $y_0^{-1}\ln(\tau/\tau_{\mathrm{ch}}) \ll 1$, the change of soliton parameters has logarithmic character and is proportional to $\ln(\tau/\tau_{\mathrm{ch}})$. In this case, the wave vector k_0, width κ^{-1}, and speed v of the soliton rise slowly, and the soliton amplitude $|\varphi|_{\max}$ decreases (see figure 4).

Let us present the computer estimates of the initial velocity v (31) and characteristic time τ_{ch} (40) for the ferromagnetic slab with the parameters referred to in [1]. Assuming that the slab thickness d is of the order of micrometres, $\omega_H \sim 10^{10}$ Hz, and $\partial_k^3\omega \sim 0.041$ (near the singular point), and taking into account that $\kappa \ll k_0$, for v and τ_{ch} we obtain respectively $v \sim 5\times10^4$ cm s^{-1} and $\tau_{\mathrm{ch}} \sim 25\times10^{18}|\Gamma|^{-2}$ s. Note that the parameter Γ is a significant variable connected with the amplitude of the wave (34) by the relation (36). Putting $C \sim 10^{-3}$ and $\varepsilon \sim 0.05$, we get $\Gamma \sim 1.6\times10^{10}$. In this case, for τ_{ch}, we have $\tau_{\mathrm{ch}} \sim 0.1$ s.

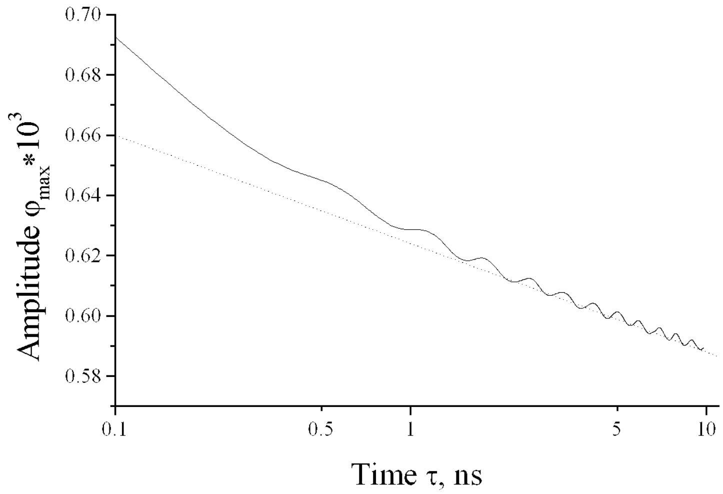

Figure 4. The changes of the soliton amplitude $\varphi_{\max}$ over time as a result of radiation (for $\kappa/k_0 = 0.05$, $\partial_k^3 \approx 4 \times 10^{-2}$, $g \approx 0.9$, and $k_0 = 10^{-1}$). The dotted curve corresponds to logarithmic asymptotics.

6. Radiation release

Another important point associated with the formation of 'bright' quasi-solitons is the question of the 'shape' of the non-linear wave sequence that is 'split off' the initial localized distribution of the magnetization. We shall take the following qualitative arguments as guidelines in the analysis of the evolution of the non-linear wave sequence. Equation (1) is contiguous to the KdV equation as regards the character of the spatial dispersion, and also to the NSE as regards the character of the non-linearity. It is known that the asymptotic behaviour of the wave sequence for $\tau \to \infty$ in the NSE and KdV models is described by the automodelling solutions [13–15]. By analogy with the KdV equation, we assume that for $\tau \to \infty$ on the space-time interval $|x| \leqslant O(\partial_k^3 \omega\, \tau)^{1/3}$ the non-linear sequence is described approximately by the following automodelling solution of equation (1):

$$\varphi = \tau^{-1/2} \chi(\eta) \qquad \eta = X\tau^{-1/3}. \tag{42}$$

The function $\chi(\eta)$ satisfies the ordinary non-linear differential equation

$$-\mathrm{i}\left[\frac{1}{2}\chi + \frac{1}{3}\eta\, \partial_\eta \chi\right] + \frac{\mathrm{i}}{6}\, \partial_k^3 \omega\, \partial_\eta^3 \chi + g|\chi|^2 \chi = 0. \tag{43}$$

The automodelling wave regime (42) developed after the stage at which the initial magnetization distribution acquired power asymptotics of the type $|\varphi|^2 \sim (-X)^{-3}$ for $X \to -\infty$ on the left-hand side (see figure 2). It is interesting that in the region $X < 0$ equation (1) has the following exact solution:

$$\varphi = \left(\frac{8}{3}\left|\frac{\partial_k^3 \omega}{g}\right|\right)^{1/2} \left(\frac{7}{4}\right)^{1/4} (-X + a)^{-3/2} \exp\left\{-\mathrm{i}\,\mathrm{sgn}(g\, \partial_k^3 \omega) \sqrt{\frac{7}{4}} \ln(-X + a) + \mathrm{i}\delta\right\}$$

where a and δ are real constants ($a > 0$). The dotted curve in figure 2 (plotted for the parameter values $\partial_k^3 \omega \approx 4 \times 10^{-2}$ and $g \approx 0.9$) corresponds to such a solution. The oscillations on the left-hand side can perhaps be interpreted as the development of the instabilities of this solution.

One cannot calculate the analytical form of the function $\chi(\eta)$, but its asymptotic behaviour can be investigated. We have found

$$\begin{aligned}
&\chi(\eta) = R_1 \exp \mathrm{i}\theta + R_2 \exp \mathrm{i}\theta + \mathrm{O}[(-\eta)^{-3/2}] \\
&\theta = \frac{2}{3}\sqrt{\frac{2}{\partial_k^3 \omega}}(-\eta)^{3/2} \\
&R_1 = R_1^0 \exp\left\{\frac{3}{2} g\, \mathrm{i} \ln(-\eta)(2|R_2^{(0)}|^2 + |R_1^{(0)}|^2)\right\} \\
&R_2 = R_2^0 \exp\left\{\frac{3}{2} g\, \mathrm{i} \ln(-\eta)(2|R_1^{(0)}|^2 + |R_2^{(0)}|^2)\right\} \qquad \eta \to -\infty \\
&\chi(\eta) = \mathrm{O}\left\{\left[\exp\left(-\frac{2}{3}\sqrt{\frac{2}{\partial_k^3\omega}}\eta^{3/2}\right)\right]\right\} \qquad \eta \to +\infty.
\end{aligned} \tag{44}$$

The complex constants $R_i^{(0)}$ $(i = 1, 2)$ in (44) acquire a dependence on the slow variable $\xi = X/\tau$ at the edge of the domain of validity of the automodelling solution. It can be shown that for $\tau \to \infty$, $X/\tau =$ constant, the automodelling solution is 'sewn together' with the following solution of equation (1):

$$\begin{aligned}
&\varphi = \sum_{m=1}^{r} \tau^{-m+1/2} \sum_{l=-\infty}^{\infty} f_{m,l}(\xi, \tau) \exp[\mathrm{i}(2l-1)\theta] \\
&\theta = \frac{\tau}{3}(\partial_k^3\omega)^{-1/2}(-2\xi)^{3/2} \\
&f_{m,l} = \exp[\mathrm{i}\tau\chi_{m,l}(\xi,\tau)]\left\{R_{m,l}^{(0,0)}(\xi) + \sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{(\ln\tau)^k}{\tau^n}R_{m,l}^{(n,k)}(\xi)\right\} \\
&\chi_{m,l}(\xi,\tau) = \sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{(\ln\tau)^k}{\tau^n}\chi_{m,l}^{(n,k)}(\xi) \\
&\xi = X/\tau \qquad \text{for } \xi < 0.
\end{aligned} \tag{45}$$

The asymptotic expansions (44) and (45) resemble the relevant expansions for the NSE and the KdV equation [13–15].

7. Conclusions

In the framework of the non-linear Shrödinger equation with dispersion of the third order, scenarios of the evolution of various initial distributions of the magnetization in thin films are analysed by analytical and computer simulation methods. It is established that, in the vicinity of the zero-dispersion point, initial spatially localized distributions of the magnetization in the first stage 'throw off' part of the energy by means of the emission of a non-linear wave sequence, and then transform into a long-living soliton-like pattern that is moving with nearly constant velocity, whose magnitude depends on the gradients of the amplitude and the phase of the initial distribution. The direction of motion of this 'quasi-soliton' is determined only by the sign of the third derivative of the dispersion law of the exchange-dipole waves. Later on, this pattern continues to radiate forward small-amplitude waves, leaving behind a quasi-static small-amplitude tail. The characteristic wave vector of the 'bright soliton', its width, and its velocity grow (logarithmically) over time, while the amplitude slowly decreases.

The process of small-amplitude modulation of stable spatially non-localized non-linear waves has been investigated also, and the possibility of the existence of 'dark solitons' has been predicted. The tendency towards the formation of 'quasi-solitons' (both 'dark' and 'bright') is quite natural under conditions of non-localized wave excitation (or during the 'throwing off' of the energy by the initial distribution of the magnetization), because the system is shifted away from the point of zero dispersion into a region where the theory of the ordinary NSE with quadratic dispersion allows the existence of 'dark' and 'bright' solitons.

It seems to us that the results obtained may have rather wide application: they are valid qualitatively also in the case in which the input equation (1) includes a term containing $\partial_k^2\omega$ as long as $\partial_k^2\omega < \partial_k^3\omega$. In that case, the equation studied can be reduced to (1) by simple identity transformations, eliminating the term involving the second derivative.

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References

[1] Kiseliev V V, Tankeyev A P and Kobelev A V 1966 *Fiz. Met. Metalloved.* **82** 38
[2] Yan H and Awai I 1977 *IEEE Trans. Magn.* **33** 2342
[3] Chen M, Tsankov M A, Nash J M and Patton C E 1993 *Phys. Rev. Lett.* **70** 1707
[4] Kodama Y and Hasegawa A 1987 *IEEE J. Quantum Electron.* **23** 510
[5] Zakharov V E and Kuznetsov E A 1988 *Zh. Eksp. Teor. Fiz.* **113** 1892
[6] Savchenko L L, Nikitov S A, Popkov A F and Chetkin M V 1998 *Zh. Eksp. Teor. Fiz.* **114** 628
[7] Gromov E M and Talanov V I 1996 *Zh. Eksp. Teor. Fiz.* **110** 137
[8] Zakharov V E, Manakov S V, Novikov S P and Pitayevsky L P 1980 *Theory of Solitons: the Inverse Problem Method* (Moskow: Nauka) p 320
[9] Wai P R A, Chen H H and Lee Y C 1990 *Phys. Rev.* A **41** 426
[10] Kuehl H H and Zhang C Y 1990 *Phys. Fluids* B **2** 889
[11] Karpman V I 1993 *Phys. Rev.* E **47** 2073
[12] Karpman V I 1998 *Phys. Lett.* A **244** 397
[13] Ablowitz M J and Segur H 1977 *Studies Appl. Math.* **57** 13
[14] Miles J W 1979 *Studies Appl. Math.* **60** 59
[15] Ablowitz M J and Segur H 1987 *Solitons and the Inverse Problem of the Scattering Method* (Moscow: Mir) p 444